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A Statistical Theory to Aggregation in One-dimensional Freeway Traffic

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Abstract The master equation of the car cluster in freeway traffic is reinvestigated on the basis of the linear assumption of the transition, in which the dynamic behavior of the cluster in a one-lane freeway traffic flow is studied. The expression of the mean size of the cluster versus t is derived in term of the solution of the birth-death equation. We also obtain the analytical expression of the probability distribution $P(n,t)$. Numerical simulations testify the results derived as well.

Keywords Traffic flow · Markov process

1 Introduction

Traffic problems have been given considerable attention of physicists since the pioneering works by Lighthill and Whitham [1–4]. There are the following reasons: traffic flow shows complex transitions and traffic states as many-body interacting systems; the fundamental principles of governing the flow of vehicular will be understood by investigating the existence of the various physical states and transitions in traffic systems, which is important to improve on the status of traffic [5].

Accordingly, varieties of traffic flow models based on mathematics and physics have been conducted during the past half century, which provide deep insight into the properties of the models and help people in better understanding the complex phenomena observed in real traffic circumstances. In general, all the traffic models may be grouped into two kinds [4]: one is macro-/mesoscopic traffic models, where the traffic flow is viewed as a compressible

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fluid formed by the vehicles but these individual vehicles do not appear explicitly in this kind [3, 6]; the other is microscopic traffic flow models focused on individual vehicles, in which cars are considered as interacting particles, such as car-following models [7] and cellular automata models [8, 9].

All traffic flow models have showed the same fundamental diagram (the flux-density diagram), which shows the phase transition from the free situation to the car cluster regime with start-stop waves. Some models have shown that for high densities a car cluster can spontaneously appear in which the average velocity of cars is considerably lower than in the initial flow. The phase transition separates the low density situation from the high-density region in which the formation of car' aggregates as bound states reduce the velocities of cars.

Some theories have been put forward to try to explain and describe these traffic jams. Barlovic et al. studied car clusters in a single-lane traffic model based on a cellular automaton approach and random walk theory, which may predict the resolving probabilities and lifetimes of jam clusters [10]. Kühne and Mahnke and others concluded the clustering behavior (known as congestion) in an initially homogeneous traffic stream by a master equation, and applied a probabilistic model to describe the traffic features [11]. Reference [12] investigated the master equation in term of the solution of the birth-death process to obtain an analytical solution to aggregation. The transition probabilities of escaping from a car cluster is considered as a constant in Refs. [11] and [12], which needs to be generalized to a general form.

In order to obtain better analytic result compared with Ref. [12], we generally construct the transition probabilities. The paper is organized as follows. In the next section, we introduce the master equation describing the car clustering behavior. We reconstruct the transition probabilities of the master equation in Sect. 3, and discuss the statistical analysis of the aggregation in Sect. 4. We make a numerical simulation in Sect. 5. Section 6 is devoted to the conclusions and discussions.

2 The Car-Cluster Model and the Master Equation

In the following, we will introduce the master equation in freeway traffic constructed by Kühne and Mahnke [1–3]. The master equation is a partial differential equation describing the probability distribution with the time evolution and random variable variation for a stochastic dynamical system that follows a Markov process. It has been successfully used to describe the linear models such as one-dimensional birth-death equations and fatigue crack growth [13–16].

Traffic flow on a single-lane and circular road is considered, and there arises a large car cluster on the road. A car cluster size n is an aggregation of n vehicles whose individual speeds are zero and the minimal allowed distances between vehicles are zero.

Assume that there appears a car cluster size m at time t, and consider that the probability of the attachment of a "free" car unit to its upstream boundary appearing in $t \to t + \triangle t$ is $W_+(m,t)\triangle t$ and the rate of cars escaping from the cluster at its downstream front is $W_-(m,t)\triangle t$. Then the probability of no change in the unit number in $t \to t + \triangle t$ is $1 - (W_+(m,t) + W_-(m,t))\triangle t$. The probability of more than one unit appearing in $t \to t + \triangle t$ is zero. Under the general assumption based on the Markov description of a birth-death process, the master equation governing the probability distribution $P(n,t)$ of the jam size n in freeway traffic is as follows:

$$\frac{\partial P(n,t)}{\partial t} = W_+(n-1,t)P(n-1,t) + W_-(n+1,t) \times P(n+1,t) - (W_+(n,t) + W_-(n,t))P(n,t), \quad (1)$$

where $P(n,t)$ is the probability to find a size-n cluster at time t. We follow the boundary conditions defined by Kühne:

$$\frac{\partial P(0,t)}{\partial t} = W_-(1,t)P(1,t) - W_+(0,t)P(0,t) \quad (2)$$

$$\frac{\partial P(N,t)}{\partial t} = W_+(N-1,t)P(N-1,t) - W_-(N,t)P(N,t) \quad (3)$$

where (2) and (3), respectively, describe the formation and dissolution of the cluster consisting of one car and the maximum possible cluster containing all the cars. Nevertheless, the transition rates (W_- and W_+) are needed in order to deal with this equation.

3 The Transition Rates

A homogeneous flow and optimal velocity model is introduced in Ref. [12], and the transition rate is constructed. We will follow and amend it. Traffic flow on a ring road is considered, and there is only the formation of a jam as a large car cluster arising on the road. We examine a circular road of length L on which N cars are moving along. All the cars are assumed to be identical vehicles of effective length l_0. The cluster is specified by its size n, the number of aggregated cars.

Because there exists only one jam on the road, some cars will transfer from the free region to the congested region, at the same time, the other cars will escape from the congested region to the low-density region. Some assumptions are necessary: 1) the relaxation for a driver to adapt himself or herself to the actual downstream state of the flow in free traffic is a time constant; 2) time constants for each car to speed up and to slow down are identical; 3) the braking time constant is much greater than the accelerating one. Thus, the headway distance h and velocity of each car in free-flow region is uniform. The headway distance h depends on the car cluster sizes n. The relaxation of the optimal velocity and the headway distance in Ref. [11] is

$$v_{opt}(h) = v_{max}\frac{h^2}{h^2 + D^2}, \quad (4)$$

where h is the headway distance (bumper-to-bumper effective distance), v_{max} is the maximal velocity, and D is a positive control parameter as the characteristic value of the heading distance at which drivers begin to feel themselves "free". The car behind the jam needs an average time h/v_{opt} if the decelerating time is zero. The growth rate of the car cluster is

$$W_+(h) = \frac{v_{opt}}{h}. \quad (5)$$

On the other hand, the function relaxation between the headway h and the car cluster size n is

$$h(n) = \frac{L - Nl_0}{N - n + 1}. \quad (6)$$

In order to obtain an analytical result, we reconstruct the growth rate based on the above hypothesis. In the free flow region, the cars are considered as noninteracting particles. If the density exceeds some critical value, a jam appears. The critical value of density is $\frac{1}{v_{max}+l_0}$, and the corresponding space headway is $v_{max}+l_0$. We assume the growth rate is linear after the jam appears. Thereby, we have

$$W_+(n) = A + Bn, \tag{7}$$

where A and B are parameters which depend on the density $k = N/L$. In order to determinate (7)'s concrete expression, we take two points $(n_{1,2}, W_+(n_{1,2}))$, i.e., $(0, W_+(0))$ and $(N - \frac{L}{v_{max}+l_0}, W_+(N - \frac{L}{v_{max}+l_0}))$. The second point is the one relative to the maximum car cluster size (all cars on the road are separated into two parts: a jam and "free" flow, and $\frac{L}{v_{max}+l_0}$ denotes the number of "free" cars that achieve the maximum velocity). The values of $W_+(0)$ and $W_+(N - \frac{L}{v_{max}+l_0})$ can be achieved by (4), (5) and (6), i.e., $W_+(0) = v_{max}\frac{h(0)}{h^2(0)+D^2}$ and $W_+(N - \frac{L}{v_{max}+l_0}) = v_{max}\frac{h(N-\frac{L}{v_{max}+l_0})}{h^2(N-\frac{L}{v_{max}+l_0})+D^2}$. When n takes zero in (7), the value of parameter A can be obtained, i.e., $A = W_+(0)$; the value of parameter B can be similarly obtained on the basis of the values of parameter A and $W_+(N - \frac{L}{v_{max}+l_0})$, then $B = \frac{W_+(N-\frac{L}{v_{max}+l_0})-W_+(0)}{N-\frac{L}{v_{max}+l_0}}$.

In earlier works, the head car of the jam needs the average time τ to come free, where τ is a constant. Thus, the corresponding transition rate of the cars escaping from the jam is simple $W_- = 1/\tau$. In reality, the transition rate of the cars escaping from the jam is influenced by other complications such as the average headway h. To extend the model, we modify the decay rate that depends on the cluster size n. We rewrite the rate of the escaping from the jam as follows

$$W_- = \alpha + \beta n, \tag{8}$$

where α and β are parameters.

4 Statistical Analysis of Aggregation

Substituting the expressions (7) and (8) into (1) yields

$$\begin{aligned}\frac{\partial P(n,t)}{\partial t} &= (A + B(n-1))P(n-1,t) + (\alpha + \beta(n+1)) \\ &\quad \times P(n+1,t) - (A + \alpha + (B+\beta)n)P(n,t).\end{aligned} \tag{9}$$

Using a generating function, we have

$$H(t,x) = \sum_{n=0}^{\infty} P(n,t)x^n, \tag{10}$$

where x is a real parameter.

Firstly, multiplying both sides of (9) by x^n and taking sum for n; secondly, with the help of (10), we have

$$\frac{\partial H}{\partial t} = \left[Ax + \frac{\alpha}{x} - (A+\alpha)\right]H + (x-1)(Bx-\beta)\frac{\partial H}{\partial x}. \tag{11}$$

Using partial differential equation theory [13–15], it follows from (11) that

$$\frac{dt}{1}=\frac{dx}{-(x-1)(Bx-\beta)}=\frac{dH}{[Ax+\frac{\alpha}{x}-(A+\alpha)]}, \tag{12}$$

which is based on the solution of partial differential equation. By solving (12), we obtain

$$\frac{x-\beta/B}{x-1}=C_1e^{(B-\beta)t} \tag{13}$$

$$C_2H=x^{-\frac{\alpha}{\beta}}\left(x-\frac{\beta}{B}\right)^{\frac{\alpha}{\beta}-\frac{A}{B}}, \tag{14}$$

where C_1 and C_2 are constants. To solve (13) and (14), we must eliminate the parameters C_1 and C_2.

Assume that the size of the cluster is n_0 at the beginning time, then we will set

$$P(n,0)=\delta_{n,n_0}, \tag{15}$$

the initial probability $P(n,0)$ is one when $n=n_0$, the others are zero.

Substituting (15) into (10), we achieve

$$H(0,x)=x^{n_0}. \tag{16}$$

When $x=1$, using the normalized condition of $P(n,t)$, we have

$$H(t,1)=\sum_{n=0}^{\infty}P(n,t)=1. \tag{17}$$

Solving (13) and (14) by the use of (17), we can obtain the analytical solution (the detailed deducing processes from (13) to (14) see Appendix A) as follows:

$$H=x^{-\frac{\alpha}{\beta}}\left(1-\frac{\beta}{B}\right)^{(\frac{A}{B}-\frac{\alpha}{\beta})}\frac{[x-\frac{\beta}{B}-\frac{\beta}{B}(x-1)e^{(B-\beta)t}]^{n_0+\frac{\alpha}{\beta}}}{[x-\frac{\beta}{B}-(x-1)e^{(B-\beta)t}]^{n_0+\frac{A}{B}}}. \tag{18}$$

Defining $-\frac{\alpha}{\beta}=m_0$, we have $W_-(n)=\beta(n-m_0)$. For convenience of discussion, we take them as integer, it follows

$$H=x^{m_0}\left(1-\frac{\beta}{B}\right)^{(\frac{A}{B}+m_0)}\frac{[x-\frac{\beta}{B}-\frac{\beta}{B}(x-1)e^{(B-\beta)t}]^{n_0-m_0}}{[x-\frac{\beta}{B}-(x-1)e^{(B-\beta)t}]^{n_0+\frac{A}{B}}}. \tag{19}$$

When $x=1$, it follows that $H(t,1)=1$, namely, (19) satisfies the condition of normalizing to unit, thus, (19) is consistent. Let $h=\frac{\beta}{B}$, we rewrite (19)

$$H=x^{m_0}(1-h)^{(\frac{A}{B}+m_0)}\frac{[x-h-h(x-1)e^{B(1-h)t}]^{n_0-m_0}}{[x-h-(x-1)e^{B(1-h)t}]^{n_0+\frac{A}{B}}}. \tag{20}$$

Let $u(x) = [x - h - h(x-1)e^{B(1-h)t}]^{n_0 - m_0}$, $v(x) = [x - h - (x-1)e^{B(1-h)t}]^{-(n_0+\frac{A}{B})}$ and $f(x) = u * v$, we have

$$H = x^{m_0}(1-h)^{(\frac{A}{B}+m_0)} f(x). \tag{21}$$

The n-order derivation of H is

$$\begin{aligned}\frac{\partial^n H}{\partial x^n}\bigg|_{x=0} &= (1-h)^{(\frac{A}{B}+m_0)} \sum_{m=0}^{n} C_n^m (x^{m_0})^m f^{(n-m)}|_{x=0} \\ &= (1-h)^{(\frac{A}{B}+m_0)} C_n^{m_0} f^{(n-m_0)} \\ &= (1-h)^{(\frac{A}{B}+m_0)} C_n^{m_0} \sum_{m'=0}^{n-m_0} C_{n-m_0}^{m'} u^{m'} (v)^{n-m_0-m'}.\end{aligned} \tag{22}$$

Using Taylor's expansion at $x=0$, we have

$$H = \sum_{n=0}^{n} \frac{1}{n!} \frac{\partial^n H}{\partial x^n}\bigg|_{x=0} x^n. \tag{23}$$

Comparing with (10), we obtain (the detailed deducing processes from (22) to (24) see Appendix B)

$$\begin{aligned}P(n,t) = {} & (1-h)^{(\frac{A}{B}+m_0)} C_n^{m_0} \sum_{m'=0}^{n-m_0} C_{n-m_0}^{m'} \{(n_0 - m_0) \\ & \times (n_0 - m_0 - 1)\cdots(n_0 - m_0 - m')(1 - he^{B(1-h)t})^{m'}\} \\ & \times (h(e^{B(1-h)t} - 1))^{n_0 - m_0 - m'} (-1)^{n_0 - m_0 - m'} \\ & \times \left(\frac{A}{B} + n_0\right)\left(\frac{A}{B} + n_0 + 1\right)\cdots\left(\frac{A}{B} + n_0 + n - m_0 \right. \\ & \left. + m' - 1\right)(1 - he^{B(1-h)t})^{n_0 - m_0 - m'} \\ & \times (e^{B(1-h)t} - h)^{-\frac{A}{B} - n_0 - n + m_0 + m'}.\end{aligned} \tag{24}$$

To have a simple expression, we define a non-integer factorial

$$(\alpha + n)_n! = (\alpha + n)(\alpha + n - 1)\cdots(\alpha + 1), \tag{25}$$

where $0 < \alpha < 1$. We obtain

$$P(n,t) = \begin{cases} \left(\frac{1-h}{e^{B(1-h)t}-h}\right)^{(\frac{A}{B}+m_0)} (h(e^{B(1-h)t}-1))^{n_0-m_0} \\ \left(\frac{1-e^{-B(1-h)t}}{1-he^{-B(1-h)t}}\right)^{n-m_0} \frac{1}{m_0!} \sum_{m'=0}^{n-m_0} \frac{1}{(n-m_0-m')!m'!} \\ \frac{(n_0-m_0)!}{(n_0-m_0-m')!} \frac{(\frac{A}{B}+n_0+(n-m_0-m')-1)_n!}{(\frac{A}{B}+n_0-1)_n}! \\ \left(\frac{h-e^{-B(1-h)t}}{h-e^{B(1-h)t}} / \frac{e^{-B(1-h)t}-1}{1-he^{-B(1-h)t}}\right)^{m'} \\ \quad \text{if } n \geq m_0, n-m_0-m' \geq 0 \\ 0, \quad \text{else.} \end{cases} \tag{26}$$

On the other hand, making using of (10) and (19), the values of the mean size of the cluster $\langle n \rangle$ and the second degree moment $\langle n^2 \rangle$ at any time are derived as follow:

$$\begin{aligned} \langle n(t) \rangle = x \frac{\partial H}{\partial x}\bigg|_{x=1} &= m_0 + (n_0 - m_0) \frac{(1 - \frac{\beta}{B} e^{(B-\beta)t})}{1 - \frac{\beta}{B}} \\ &\quad - \left(n_0 + \frac{A}{B}\right) \frac{1 - e^{(B-\beta)t}}{1 - \frac{\beta}{B}} \end{aligned} \tag{27}$$

$$\begin{aligned} \langle n^2(t) \rangle = x \frac{\partial}{\partial x}\left(x \frac{\partial H}{\partial x}\right)\bigg|_{x=1} &= {m_0}^2 + (n_0 + m_0(n_0 - m_0)) \\ &\quad \times \frac{1 - \frac{\beta}{B} e^{(B-\beta)t}}{1 - \frac{\beta}{B}} - \left(n_0 + \frac{A}{B}\right)(2m_0 + 1) \frac{1 - e^{(B-\beta)t}}{1 - \frac{\beta}{B}} \\ &\quad - 2(n_0 - m_0)\left(n_0 + \frac{A}{B}\right) \frac{(1 - \frac{\beta}{B} e^{(B-\beta)t})(1 - e^{(B-\beta)t})}{(1 - \frac{\beta}{B})^2} \\ &\quad + (n_0 - m_0)(n_0 - m_0 - 1) \frac{(1 - \frac{\beta}{B} e^{(B-\beta)t})^2}{(1 - \frac{\beta}{B})^2} \\ &\quad \times \left(n_0 + \frac{A}{B}\right)\left(n_0 + \frac{A}{B} + 1\right) \frac{(1 - e^{(B-\beta)t})^2}{(1 - \frac{\beta}{B})^2} \end{aligned} \tag{28}$$

On the assumption that the growth rate of the cluster and escaping from the cluster are linear and in terms of the solution of a birth-death process, we derive some important results. Equation (27) describes the dynamic behavior of the car cluster far from critical densities of the initial size. To study the distributing character of random variables $P(n,t)$, we obtain the solution of the probability density function $P(n,t)$. We will choose reasonable parameters based on realistic flow-density behavior and simulate the dynamic behavior in the following section.

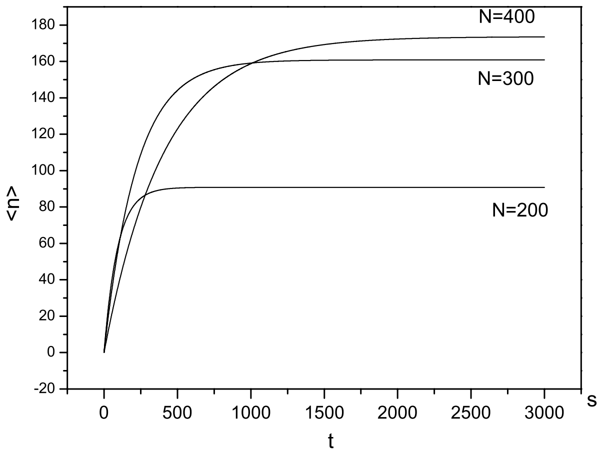

Fig. 1 The curves of the mean cluster size $\langle n \rangle$ starting with as an initially homogeneous traffic steam at different densities versus time t

5 Numerical Results

A car cluster arises which depends on the rates of W_+ and W_-. In a low density range, a car cluster does not occur and cars are noninteracting; there is a car cluster as soon as the condition of $W_+ > W_-$ is satisfied. Thus when the growing rate is always less than the decay rate, any initial cluster will disappear. Once $W_+ > W_-$, a cluster will appear on the road. When the growing transition balances with the decay transition, a stable stationary size of the cluster will occurs, which means that the system will attain to equilibrium.

Considering the linear assumption about the rates, there exists only one equilibrium size for $W_+ = W_-$.

We choose the same parameter values as Ref. [12] which are based on realistic data so that we can get realistical analytical results. The parameters are $l_0 = 8$ m, $u_f = 40$ m/s, $D = 20$ and $L = 5000$. However, it is difficult to obtain some parameter values such as α and β. In earlier works, the head car of the car cluster that comes free is assumed to need the average time τ which is constant. In Ref. [12], the value of τ is 7 seconds. To approach the value, we make $\alpha = 7$ and $\beta = -\frac{4}{N_{max}}$, where N_{max} indicates the maximal number of cars on the road.

Equation (27) describes the dynamics behavior of the system which is far from $W_+ = W_-$. We simulate the dynamics behavior of the car cluster. Figure 1 shows the curves of the mean cluster size versus time t at different values of car densities. On the other hand, for different values of initial n_0, the system finally attain the same steady state (see Fig. 2). To illustrate $P(n,t)$ evolution with time, the time evolution of probability $P(n,t)$ has been simulated for a finite size 150 cars moving along on the road (see Fig. 3). The probability distribution of $P(n,t)$ at time steps starting with $P(10,0) = 1$ as initial condition is obtained. It makes clear that the probability distribution tends to an approximate normal distribution with time evolution. Numerical simulations accord with the derived results very well.

6 Conclusion and Summary

To obtain a general form, we extended the master-equation approach to the case of congestion in traffic flow based on the linear assumption of the transition rates. The growth rate of the attachment of “free” cars to the cluster and the rate of escaping from the cluster all

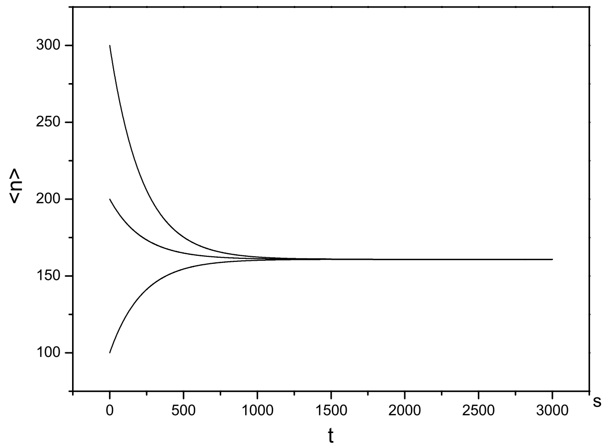

Fig. 2 The curves of the mean cluster size $\langle n \rangle$ versus time t at different values of n_0 for the same car density ($N = 300$)

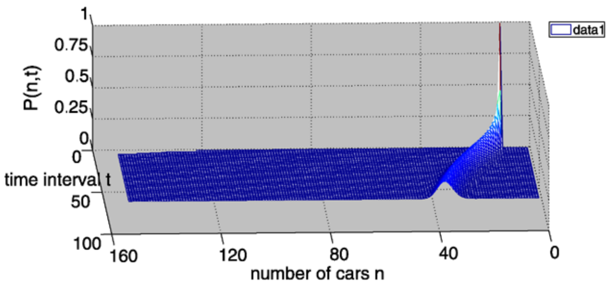

Fig. 3 The figure show that the probability distribution $P(n,t)$ evolves with time starting with $P(10,0) = 1$ as initial condition, where $N = 150$ and $n_0 = 10$

depend on the cluster size n. The linear assumption will lead to existing an equilibrium size over critical densities. Consulting the solution of a birth-death process, we derive the values of the cluster mean size $\langle n \rangle$ and the second degree moment $\langle n^2 \rangle$ at any time t. We make the analytic result of the probability distribution. In parallel, we simulate the dynamic behavior of the cluster and obtain the curves of the cluster mean size $\langle n \rangle$ versus time t starting with an initial homogeneous traffic flow at different densities (see Fig. 1) and different values of n_0 (see Fig. 2). Furthermore, the figure of the probability distribution $P(n,t)$ evolving with time and n is obtained, which tends to an approximate normal distribution with time evolution. Numerical simulations accord with the derived results very well.

In traffic models, traffic jam is paid more attention by traffic researchers. The results presented by us will contribute to forecasting traffic jam and a car cluster size. In future work, we will focus on the breakdown of traffic bottlenecks based on our present methods.

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Appendix A

We will deduce from (13) to (18) in detail. Since we must eliminate the parameters C_1 and C_2, we assume

$$C_1 = \varphi(C_2). \tag{A.1}$$

When $t=0$, though (13), we have

$$x=\frac{C_1-\frac{\beta}{B}}{C_1-1} \tag{A.2}$$

and

$$H(0,x)=x^{n_0}. \tag{A.3}$$

Substituting (A.2) and (A.3) into (14), we achieve

$$\begin{aligned} C_2&=x^{-\frac{\alpha}{\beta}}\left(x-\frac{\beta}{B}\right)^{(\frac{\alpha}{\beta}-\frac{A}{B})}\Big/H(0,x)\\ &=x^{-\frac{\alpha}{\beta}}\left(x-\frac{\beta}{B}\right)^{(\frac{\alpha}{\beta}-\frac{A}{B})}\Big/x^{n_0}\\ &=\left(\frac{C_1-\frac{\beta}{B}}{C_1-1}\right)^{-\frac{\alpha}{\beta}-n_0}\left(\frac{C_1-\frac{\beta}{B}}{C_1-1}-\frac{\beta}{B}\right)^{(\frac{\alpha}{\beta}-\frac{A}{B})}\\ &=\varphi(C_1), \end{aligned} \tag{A.4}$$

where we obtain the function expression $\varphi(C_1)$. By (14), we know

$$C_2=x^{-\frac{\alpha}{\beta}}\left(x-\frac{\beta}{B}\right)^{\frac{\alpha}{\beta}-\frac{A}{B}}\Big/H(t,x)=\varphi(C_1). \tag{A.5}$$

Thus,

$$\begin{aligned} H(t,x)&=x^{-\frac{\alpha}{\beta}}\left(x-\frac{\beta}{B}\right)^{\frac{\alpha}{\beta}-\frac{A}{B}}\Big/\varphi(C_1)\\ &=x^{-\frac{\alpha}{\beta}}\left(x-\frac{\beta}{B}\right)^{\frac{\alpha}{\beta}-\frac{A}{B}}\Big/\varphi\left(\left(\frac{x-\frac{\beta}{B}}{x-1}\right)e^{-(B-\beta)t}\right)\\ &=x^{-\frac{\alpha}{\beta}}\left(x-\frac{\beta}{B}\right)^{\frac{\alpha}{\beta}-\frac{A}{B}}\Big/\left[\left(\frac{((\frac{x-\frac{\beta}{B}}{x-1})e^{-(B-\beta)t})-\frac{\beta}{B}}{((\frac{x-\frac{\beta}{B}}{x-1})e^{-(B-\beta)t})-1}\right)^{-\frac{\alpha}{\beta}-n_0}\right.\\ &\quad\left.\times\left(\frac{((\frac{x-\frac{\beta}{B}}{x-1})e^{-(B-\beta)t})-\frac{\beta}{B}}{((\frac{x-\frac{\beta}{B}}{x-1})e^{-(B-\beta)t})-1}-\frac{\beta}{B}\right)^{(\frac{\alpha}{\beta}-\frac{A}{B})}\right]. \end{aligned} \tag{A.6}$$

By simplify, we obtain (18).

Appendix B

In (22), we have

$$\begin{aligned} u^m(0,t)&=(n_0-m_0)(n_0-m_0-1)\cdots(n_0-m_0-(m-1))\\ &\quad\times(1-he^{B(1-h)t})^m[h(e^{B(1-h)t}-1)]^{n_0-m_0-m} \end{aligned}$$

$$= \frac{(n_0 - m_0)!}{(n_0 - m_0 - m)!}(1 - he^{B(1-h)t})^m \times [h(e^{B(1-h)t} - 1)]^{n_0 - m_0 - m} \tag{B.1}$$

and

$$\begin{aligned} v^m(0,t) &= \{[(e^{B(1-h)t} - h) + x(1 - e^{B(1-h)t})]\}^m \\ &= \left(-\frac{A}{B} - n_0\right)\left(-\frac{A}{B} - n_0 - 1\right)\cdots\left(-\frac{A}{B} - n_0 - m + 1\right) \\ &\quad \times (1 - e^{B(1-h)t})^m (e^{B(1-h)t} - h)^{-\frac{A}{B} - n_0 - m}. \end{aligned} \tag{B.2}$$

To have a simple expression, we define a non-integer factorial

$$(\alpha + n)_n! = (\alpha + n)(\alpha + n - 1)\cdots(\alpha + 1), \tag{B.3}$$

Then we obtain

$$v^m(0,t) = (-1)^m \frac{(\frac{A}{B} + n_0 + m - 1)_n!}{(\frac{A}{B} + n_0 - 1)_n!} \times (1 - e^{B(1-h)t})^m (e^{B(1-h)t} - h)^{-\frac{A}{B} - n_0 - m}. \tag{B.4}$$

By compare (10) with (23), we have

$$P(n,t) = \frac{1}{n!}\frac{\partial^n H}{\partial x^n}\bigg|_{x=0}. \tag{B.5}$$

Thus, we obtain

$$\begin{aligned} P(n,t) &= \frac{1}{n!}\frac{\partial^n H}{\partial x^n}\bigg|_{x=0} \\ &= \frac{1}{n!}(1-h)^{\frac{A}{B}+m_0} C_n^{m_0} \sum_{m'=0}^{n-m_0} C_{n-m_0}^{m'} u^{m'} v^{n-m_0-m'} \\ &= (1-h)^{(\frac{A}{B}+m_0)} C_n^{m_0} \sum_{m'=0}^{n-m_0} C_{n-m_0}^{m'} \{(n_0 - m_0) \\ &\quad \times (n_0 - m_0 - 1)\cdots(n_0 - m_0 - m')(1 - he^{B(1-h)t})^{m'} \\ &\quad \times (h(e^{B(1-h)t} - 1))^{n_0 - m_0 - m'} (-1)^{n_0 - m_0 - m'}\} \\ &\quad \times \left(\frac{A}{B} + n_0\right)\left(\frac{A}{B} + n_0 + 1\right)\cdots\left(\frac{A}{B} + n_0 + n - m_0 \right. \\ &\quad \left. + m' - 1\right)(1 - he^{B(1-h)t})^{n_0 - m_0 - m'} \\ &\quad \times (e^{B(1-h)t} - h)^{-\frac{A}{B} - n_0 - n + m_0 + m'}. \end{aligned} \tag{B.6}$$

By simplify, we finally obtain (26).

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